SCOPE OF SIMULATION OF GROUNDWATER FLOW USING MESHLESS METHOD, AN ALTERNATIVE NUMERICAL METHOD

Boddula Swathi[1] and Eldho, T.I[2]

Research Scholar, Department of Civil Engineering, IIT Bombay, boddulaswathi@gmail.com
[2] Dr, Professor, Department of Civil Engineering, IIT Bombay, eldho@civil.iitb.ac.in

Abstract: *Groundwater constitutes an important component of water resources systems. As it is scarcer than the surface water, it has to be properly managed for sustainable use. Good management requires the ability to forecast the aquifer system's response under different scenarios. For this, model with set of governing equations are solved using analytical or numerical methods. World wide now, conventional numerical methods like Finite Difference Method (FDM) and Finite Element Method (FEM) are being used for simulation of groundwater flow. However, they have inherent shortcomings of relying on meshes/grids or elements that are connected together by nodes in a properly predefined manner requiring huge efforts in pre-processing. Over the last decade, intense research is going on alternative numerical approach called Meshless methods, overcoming the limitations. In this paper, different Meshless methods are introduced and works done using these methods in Groundwater are reviewed. Also, a simulation model is developed with Meshless Local Petrov-Galerkin (MLPG) using Exponential/Guassian Radial basis function (EXP-RBF). A computer model in MATLAB has been developed for 1D and 2D cases for Groundwater flow. The developed 1D model is verified with available analytical solution for a hypothetical problem. Further the 2D model is applied to a case study and results are compared with the available FEM solution and found to be satisfactory. This technique gave quiet promising results showing the applicability of this approach in groundwater flow simulation.*

Keywords: *Groundwater flow, Meshless method, Meshless Local Petrov-Galerkin (MLPG) method, Gaussian Radial Basis Function*

1. INTRODUCTION

Groundwater constitutes an important component of many water resource systems. As availability of groundwater is a scarce compared to the surface water, it should be used very effectively. But in view of the increasing population and uncontrolled pollution worldwide, there is an urgent need to manage and protect these shrinking groundwater resources efficiently to ensure its sustainable utilization. As groundwater process being insidious and complex in nature, it is important to predict the distribution of groundwater in an aquifer with space and time with respect to pumping operations. For this, modelling plays a pivotal role in the design and management of groundwater systems.

The groundwater flow is generally described by a partial differential equation, which commonly been solved now-a-days using numerical methods like Finite Difference Method (FDM) and Finite Element Method (FEM) (Pinder and Gray, 1977). However, they have inherent shortcomings of relying on meshes/grids or elements that are connected together by nodes in a properly predefined manner. High costs in creating an FDM/FEM mesh, low accuracy of prediction,

difficulty in adaptive analysis are few shortcomings of these methods. The root of these problems is the use of elements or mesh in the formulation stage. The idea of getting rid of meshes in the process of numerical treatments has naturally evolved and the concepts of Meshless Methods have been shaped up.

Meshless method (Liu, 2002) is best described as a numerical method used to establish a system of algebraic equations for the whole problem domain without the use of a predefined mesh for the domain discretization. Meshless methods use a set of nodes scattered within the problem domain as well as set of nodes on the boundaries of the domain to represent (not discretize) the problem domain and its boundaries. As these nodes do not form a mesh and thereby they do not require prior information on the relationship between the nodes for the interpolation or approximation of the unknown functions of the field variables.

Due to the tedious meshing or re-meshing in problems of crack propagation, shear band formation, large deformations, element distortion and locking (shear, membrane, incompressibility etc.,) made the Civil engineers to look for methods, which can solve these meshing problems.

Many Meshless methods have found good applications and shown very good potential to become powerful numerical tools in various fields of engineering (Liu, 2002). Some of the Meshless methods developed with their features are given in Table 1.However, the Meshless methods are still in developing stage, and there are technical problems that need to be resolved before the methods can become efficient tools for complex engineering problems.

Table 1 Meshless Methods (Modified from Liu, 2002)

Method	Reference	System Equation to be solved	Method of Function Approximation
Diffuse Element Method	Nayroles et al., 1992	Weak form	MLS, Galerkin
Element Free Galerkin	Belytschko et al., 1994	Weak form	MLS, Galerkin
Meshless Local Petrov Galerkin Method	Atluri and Zhu, 1998	Local Weak form	MLS, Galerkin
Finite Point Method	Onate et al., 1996	Strong form	Taylor series, MLS
Smooth Particle Hydrodynamics	Lucy, 1977; Gingold and Monaghan, 1977	Strong form	Integral representation
Reproducing Kernel Particle Method	Liu, W.K. et al., 1993	Strong or weak form	Integral representation
hp- clouds	Oden and Abani, 1994	Weak form	PU, MLS
Partition of unity Finite Element Method	Babuska and Melenk, 1995	Weak form	PU, MLS
Point Interpolation Method	Liu, G.R and Gu, 1999	Weak form and local weak form	Point interpolation
Boundary Node Method	Mukherjee and Mukherjee, 1997	Weak form and local weak form	MLS
Boundary Integral Equation Method	Zhu et al., 1998	Weak form and local weak form	MLS

(***Note***: MLS – Moving Least Square; PU - Partition of Unity)

In groundwater studies, Li, et al (2003) used Collocation method with Radial basis function for groundwater contaminant transport modelling. Praveen Kumar and Dodagoudar (2008) used Radial Point interpolation method (RPIM) to simulate contaminant transport through saturated porous media. Saeedpanah et al (2011) used local radial point interpolation meshless (LRPIM)

method to study the effect of leakage on tidal response in a coastal leaky confined aquifer system. Zi Li and Xian Zhong Mao (2011) used radial basis collocation method (RBCM) for the inverse problem of groundwater contaminant source identification. Mategonkar and Eldho (2012) used Polynomial Point Collocation method (PPCM) for Simulation- Optimization of groundwater contamination remediation. Kovářík et al (2012) used Meshless Radial Basis Integral Equation (RBIE) method to solve the unsteady groundwater flow.

In this paper, Meshless local weak form method, Meshless Local Petrov-Galerkin (MLPG) method with Exponential/Guassian Radial basis function (EXP-RBF) is used for solving the groundwater flow equation. The objective of the study is to develop Meshless based computer model in MATLAB, for a confined groundwater flow equation, test its adequacy and to compare the results with available solutions and show the effectiveness of Meshless method.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The governing equation describing the flow in a two dimensional confined aquifer is given as (Bear, 1979):

(1)

Following initial conditions are used for transient flow analysis,

$h(x, y,0) = h_o(x, y)$ Ω *(2)*

Generally, the boundary conditions can be of two types: the prescribed head (Dirichlet boundary condition) or known flux (Neumann boundary condition). It can be written as:

$h(x, y, t) = h_1(x, y, t)$ Ω_1

Ω_2 *(3)*

Here, $h(x, y, t)$ = Piezometric head (m), $T(x, y)$ = Transimissivity (m^2/d), S = Storage coefficient, x, y = Horizontal space variables (m), Q_w = Source or sink function ($-Q_w$ source, Q_w = Sink) ($m^3/d/m^2$), t = Time in days, Ω = The flow region , $\partial\Omega$ = The boundary region ($\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$) , $\frac{\partial}{\partial n}$ = Normal derivative, $h_0(x, y)$ = Initial head in the flow domain (m), $h_1(x, y, t)$ = Known head value of the boundary head (m), $q(x, y, t)$ = Known inflow rate ($m^3/d/m$), δ is dirac delta function, δ = 1 if $x = x_i, y = y_i$ or δ = 0 if $x \neq x_i, y \neq y_i$

3. MLPG FORMULATION

The transient groundwater flow for homogeneous confined isotropic media in 2D can be simplified and written as (Bear, 1979):

(4)

Where, *S* is the storativity and *T* is the transmissivity of the aquifer. In MLPG formulation, (Atluri, 2005) the first step is to define the trial solution, where, h_i *is* the unknown head; R_i *(x, y)* is the shape function at node *i* and n is the total number of nodes in the support domain. In the present study, the shape function R_i *(x. y)* is taken as (Powell, 1992)

(5)

Where, *and Cs =,* is the dimensionless size of support domain, and d_c is the nodal spacing near the node *i,* x_i and y_i are nodal co-ordinates.

As the approximate solution does not satisfy the equation, a residual error term results. The MLPG method requires that the weighting function and residual product integral is made equal to zero. Using MLPG principle of local residual formulation (Atluri, 2005), the equation (4) can be written as follows.

(6)

Using integration by parts, the first term can be converted as

(7)

The second term can also be written in similar way. Hence eq (6) can be written as

(8)

Last term in eq (8) is flux term normal to the boundary which can be represented with the help of given boundary condition. If the boundary is impervious, the flux normal to the boundary is zero, otherwise the term can be written as,

(9)

Where q is the known flux

3.1 Time domain discretization

Based upon the numerical experiments conducted by Pinder and Gray (1977) and by Wang and Anderson (1982), it is concluded that the finite difference scheme in time is the best overall choice in the majority of transient analysis.

Hence, to account for the time derivative, following forward finite difference approximation is used in this study. i.e.,

(10)

In this study, fully implicit scheme is used in the model development. The solution procedure begins with the specified initial values of [h] at t=to. Then the system is solved to obtain the values of [h]at the end of time step and the process is repeated for the next time step and so on.

Using the finite forward difference scheme for time derivatives, eq (8) can be written as

(11)

Since *, eq.(11) can be written as*

(12)

From eq (12), the system of equations can be written in the matrix form as

(13)

Where *is a global matrix for square of shape function, is global matrix for square of first derivative of shape function with respect to x and is global matrix for square of first derivative of shape function with respect to y, such that*

(14)

(15)

(16)

Here, the shape function is written as *(17)*

The*, and are calculated for every support domain initially and global matrices later on are formed and the* Eq.(13) is solved using the Gauss-Jordan method.

When the transient groundwater flow equation for heterogeneous confined anisotropic media in 2D is considered, the governing equation can be written as (Bear, 1979):

(18)

The MLPG formulation for this equation is similar to the homogeneous aquifer as explained earlier except that T is replaced by T_x and T_y, where T_x and T_y are the transmissibilities in x and y directions, respectively. For the heterogeneous media, the domain is divided into zones and the transmissibility of that zone is considered for all the nodes lying in that particular zone.

When the confined anisotropic media is considered, the final form of Eq. (13) can be written as:

(19)

When source or sink terms are to be considered, then Eq. (13) can be written as:

(20)

Where, *;* q_w is the source or sink term (m^3/day) ; 'a' is the area of support domain in which the pumping well or recharge well lies (m^2) and Q_w is the global matrix of the entire source and sink terms. The pumping and recharge wells are considered for source and sink terms.

4. MODEL DEVELOPMENT

Based on the MLPG formulation, as given above, 1D and 2D flow models for confined aquifer (MLPG-GFMC-1D and MLPG-GFMC-2D) are developed for transient conditions. For the given problem, entire domain is represented using nodes. Here, depending on the problem, equidistant nodes or irregular nodes may be used which are added in the aquifer domain and on the boundary. The value of shape parameter, for 2D cases, α_c (dimensionless size of the support domain) should have the value between 2.5 and 3.5 (Liu, 2002) for accuracy as well as stability of the results. Hence for present study, square support domain is considered such

that the dimension of support size, α_c will be 3 in all directions as shown in Fig 1. The following are the brief steps to be followed.

1. Domain Representation with nodes
2. Function Approximation of the unknown
3. Formation of System of Equations
4. Solving Global Equations

5. MODEL VERIFICATION FOR MLPG-GFMC-1D

For the model MLPG-GFMC-1D verification, flow in 1D homogeneous, confined and isotropic medium is considered and compared with the analytical solution for confined aquifer problem given by Ligget and Liu (1983). The diagrammatic representation of the problem domain is shown in Fig 2.

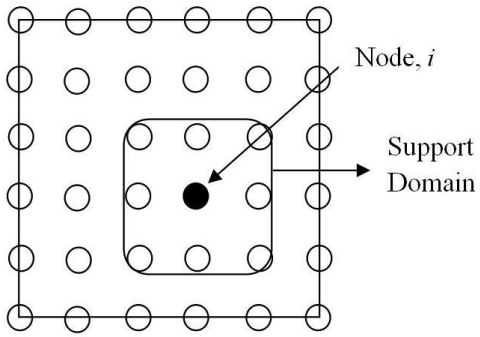


Fig.1. Local Support domain used in 2D cases to construct shape function

The problem domain is divided into 11 nodes with biased 7 nodes in each support domain as shown in Fig. 3. The model is applied to the above problem using the value of shape parameters as *and therefore, c= 0.07.* A time step of 1 min is taken. Total time of simulation is taken as 1000 min.

The simulation results by model are compared with an analytical solution given by Ligget and Liu (1983) and Meshless model using Point collocation Method (PCM) (Mategaonkar, 2011) and are shown in Table 2.The analytical solution as given in Ligget and Liu (1983) for the above problem is,

(21)

Where, x is the distance from origin, n= 1, 2, 3...and t is the time.

Table 2 shows the comparison of the present MLPG results with the analytical solution and another Meshless model using Point Collocation Method (PCM) by Mategaonkar and Eldho (2011). Good agreement is observed between the results showing the applicability of the MLPG based model. The MLPG model results are found to be more accurate than PCM results.

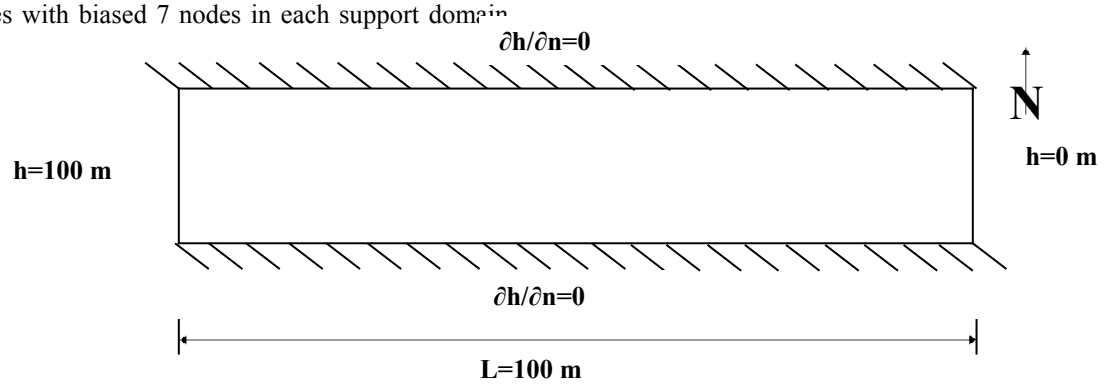


Fig. 2: Diagrammatic representation of aquifer for 1D flow problem (confined)

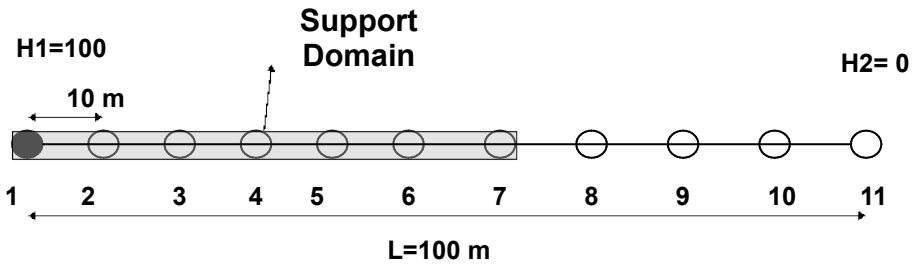


Fig. 3: Nodal representation for 1D flow problem (confined)

Table 2: Head distribution in m by analytical solution, PCM solution, MLPG solution at t=1000 days with Δt=0.1 day

Node No.	X (in m)	Analytical	PCM	% Difference Analytical-PCM	MLPG	% Difference Analytical-MLPG
1	0	100	100	0	100	0
2	10	90	90.27789	0.3088	89.95095	0.0545
3	20	80	80.23806	0.2976	79.95708	0.0536
4	30	70	70.2382	0.3403	69.96321	0.0526
5	40	60	60.19909	0.3318	59.96933	0.0511

6. MODEL VERIFICATION FOR MLPG-GFMC-2D

The MLPG-GFMC-2D model is applied to a hypothetical confined aquifer case study for the simulation of 2D flow. The configuration of the aquifer considered is shown in the Fig. 4. The dimension of the aquifer is 1400 m x 1400 m with the pumping well at the centre and it is surrounded by impervious boundaries (Willis and Yeh, 1987).

The transmissivity and storage coefficient of the aquifer are 100 m^2/day and 0.001, respectively. The aquifer is initially at a steady state condition with piezometric head equal to 100 m throughout the aquifer. A pumping well, extracting 10000 m^3/day is located at the centre of the square aquifer, node at 113. Observation wells are located at nodes 29, 43, 57, 71, 85 and 99 (Fig. 5). The Western and Eastern boundaries are constant head boundaries with values of 100 m and 100 m, respectively. Northern and Southern boundaries are no flow boundaries. For MLPG-GFMC-2D model, 225 equidistant nodes are considered and 9 nodes are considered in every support domain. The nodal arrangement is shown in the Fig. 5. Δx=Δy=100m and value of *and therefore, c= 0.0003.*

The analysis is done with 0.2 day time step with total simulation time of 10 days. In the present study, simulation results by MLPG-GFMC-2D model is compared with with the analytical solution, as given by Chan et al. (1976), FEM simulation results and Meshfree method using PCM (Mategaonkar, 2011). The comparison for 10 days simulation is given in the Table 3. As can be seen from Table 3 the MLPG-GFMC-2D model prediction is better than FEM and PCM. The head variation in the aqufier in the form of contour is showed in the Fig. 6.

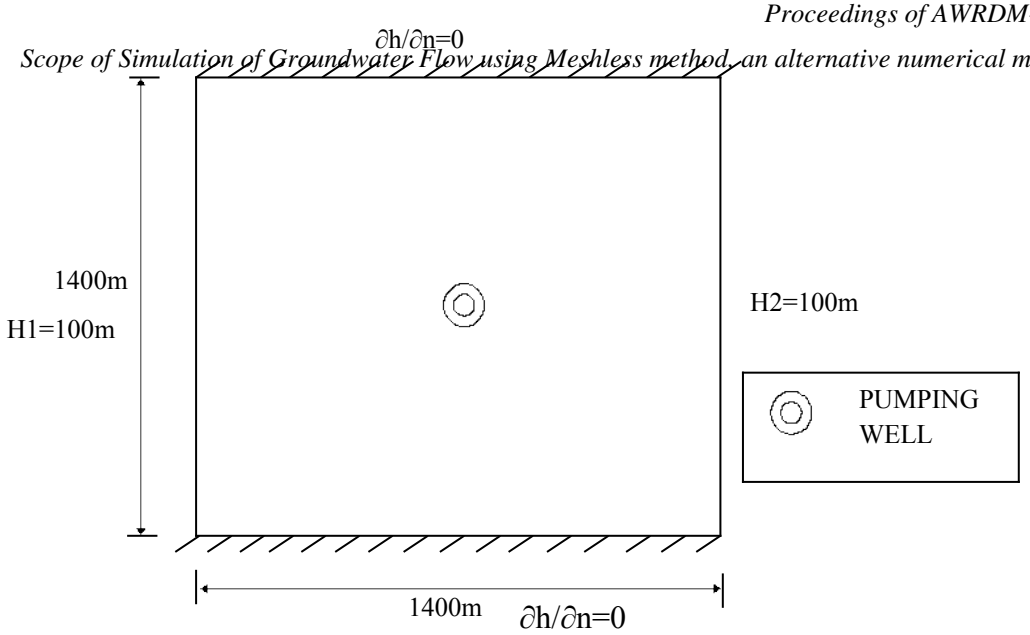


Fig. 4: **Aquifer configuration for 2D flow problem**

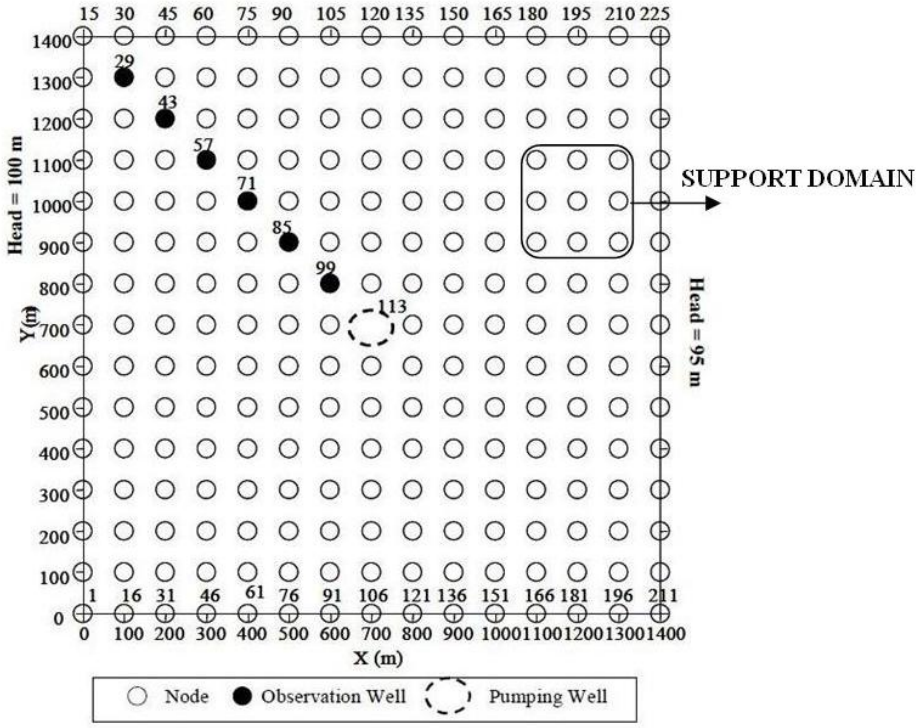


Fig. 5: Nodal arrangement for 2D (confined) flow problem

Table 3: Head distribution in m by analytical solution, FEM Solution, PCM solution, MLPG solution at t=10 days with Δt=0.2 day

Node No.	Analytical	FEM	%Diff Analy-FEM	PCM	%Diff Analy-PCM	MLPG	%Diff Analy-MLPG
15	100	100	0	100	0	100	0
29	97.013	96.993	0.021	97.0291	0.017	97.1282	0.119
43	93.804	93.768	0.038	93.7799	0.026	93.7453	0.063

57	90.095	90.051	0.049	90.0713	0.026	90.1255	0.034
71	85.451	85.413	0.044	85.2667	0.216	85.4758	0.029
85	78.983	78.974	0.011	78.9133	0.088	78.9762	0.009
99	67.953	67.762	0.281	67.6195	0.491	67.8768	0.112

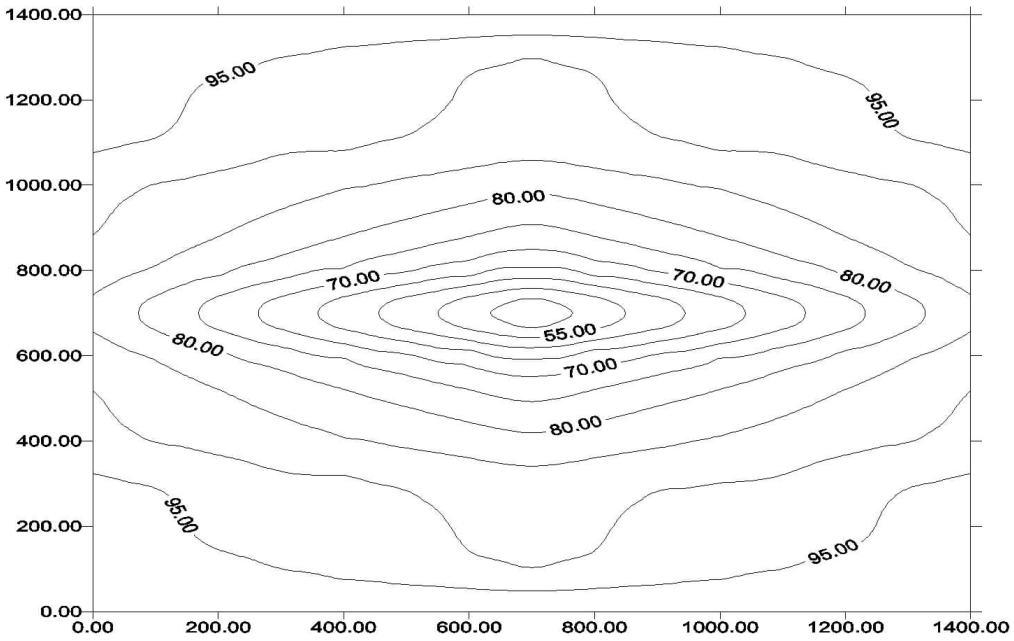


Fig. 6: Head distribution Contour for 2D flow problem

7. COMPUTATIONAL TIME- COMPARISON BETWEEN FEM AND MESHLESS MLPG AND MESHLESS PCM AND DISCUSSION

MLPG model gives good results for simulation of groundwater flow and transport. Though preprocessing time is saved in Meshless, it is required to see whether the computational time is also saved as compared to FEM. For this study, consider the MLPG-GFMC-2D model example. For FEM simulation, 196 nodes and 392 elements are considered while for MLPG-GFMC-2D model and PCM-GFMC-2D (Mategaonkar, 2011), 225 nodes and 9 nodes in every support domain are considered. The time step is of 0.2 day is considered.

The simulation is carried out in a computer with Core i5 processor with speed of 3.1 Ghz and with 4 Gb RAM, the ground water heads are obtained using FEM in time of 1.232 seconds and a comparatively lesser time of 0.861 seconds for Meshless PPCM method and a time of 1.264 seconds for MLPG-GFMC-2D. This proves that the computational time required for Meshless methods is substantially less. But in case of MLPG, it is computationally more expensive if the same field nodes are used. This mainly comes from the numerical integrations, and solving the asymmetric stiffness matrix. The great challenge is to improve the speed and robustness of the method, to match the corresponding computational levels of FEM. However the time in pre-processing of data for MLPG is much less compared to FEM, as we do not need to construct a mesh. Improvements on above issues for MLPG is very essential and there is high scope of research in this field and are considered in further research.

8. CONCLUDING REMARKS

In this paper, a Meshless Meshless Local Petrov-Galerkin (MLPG) method is presented for

groundwater flow simulation. The developed MLPG model is verified with available analytical solution and FEM solution and found to be satisfactory. From the results, it can be observed that the developed MLPG model can be effectively used in the simulation of groundwater flows. However, establishment of support domain, choice of distribution of nodes and the parameter, C*s* values are still issues which have to be studied further. In further studies, the trial/shape and test functions will be taken from different function spaces and will be compared.

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